# An application of Rouché's theorem

**Statement of the theorem:**
Suppose $f$ and $g$ are holomorphic in an open set containing a closed, simple curve $\gamma$ and its interior. If $|f(z)| > |g(z)|$ for all $z \in \gamma$, then $f$ and $f + g$ have the same number of zeros in the interior of $\gamma$

**An application:**
Let $f(z) = 2z^5 + 8z - 1$. All five zeros of $f(z)$ are inside the disc $|z| < 2$ and exactly one zero is inside the disc $|z| < 1$

Proof: Let $g(z) = 2z^5$ and let $h(z) = 8z - 1$. For $|z| = 2$, $|g(z)| = |2z^5| = 2|2^5| = 64 > 17 = 8|2| + 1$ $= |8z| + |-1| \geq |8z - 1| = |h(z)|$. By Rouché's theorem, the number of zeroes of $g(z)$ inside the disc $|z| < 2$ (five, with multiplicity) equals the number of zeros of $g(z) + h(z) = f(z)$

For $|z| = 1$, $|h(z)| = |8z - 1| \geq |8z| - |1| = 7 > 2 = 2|z^5| = |g(z)|$. Again, by Rouché's theorem, the number of zeroes of $h(z)$ inside the disc $|z| < 1$ (one) equals the number of zeros of $g(z) + h(z) = f(z)$. $\square$